CONVEX BODIES OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

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1. Introduction.

A *convex body* in the n-dimensional Euclidean space $\mathbf{R}^n$ is a compact convex set with non-empty interior. A convex body K in three dimensional Euclidean space has *constant width* w iff the orthogonal projection of K onto every line is an interval of length w. It has *constant brightness* b iff the orthogonal projection of K onto every plane is a region of area b.

Theorem 1. *Any convex body in* $\mathbf{R}^3$ *of constant width and constant brightness is a Euclidean ball.*

Under the extra assumption that the boundary is of class C^2 this was proven by S. Nakajima (= A. Matsumura) [18] in 1926 (versions of Nakajima's proof can be found in the books of Bonnesen and Fenchel [3, Sec. 68] and Gardner [7, p. 117]). Since then the problem of determining if there is a non-smooth non-spherical convex body in $\mathbf{R}^3$ of constant width and constant brightness has become well known among geometers studying convexity (cf. [5, p. 992], [7, Prob. 3.9 p. 119], [8, Ques. 2, p. 437], [10, p. 368]). Theorem 1 solves this problem.

For convex bodies with C^2 boundaries and positive curvature Nakajima's result was generalized by Chakerian [4] in 1967 to "relative geometry" where the width and brightness are measured with with respect to some convex body K_0 symmetric about the origin called the *gauge body*. While the main result of this paper is Theorem 1, Chakerian's methods generalize and simplify parts of our original proof. The following isolates the properties required of the gauge body. Recall the *Minkowski sum* of two subsets A and B of $\mathbf{R}^n$ is $A + B = \{a + b : a \in A, b \in B\}$.

Definition. A convex body K_0 is a *regular gauge* iff it is centrally symmetric about the origin and there are convex sets K_1, K_2 and Euclidean balls B_r and B_R such that $K_0 = K_1 + B_r$ and $B_R = K_0 + K_2$.

Any convex body symmetric about the origin with C^2 boundary and positive Gaussian curvature is a regular gauge (Corollary 2.4 below). For any linear subspace P of $\mathbf{R}^n$ let $K|P$ be the projection of K onto P (all projections in this paper are orthogonal). For a unit vector u let $w_K(u)$ be the width in the direction of u. For each positive integer k and any Borel subset A of $\mathbf{R}^n$ let be $V_k(A)$ be the k-dimensional volume of A (which in this paper is the k-dimensional Hausdorff measure of A). Two subsets A and B of $\mathbf{R}^n$ are *homothetic* iff there is a positive scalar λ and a vector v_0 such that $B = v_0 + \lambda A$.

Date: June 30, 2003.
2000 *Mathematics Subject Classification.* Primary 52A15, Secondary 52A20, 52A40, 30C62 .
Key words and phrases. Constant width, constant brightness, relative geometry, quasiconformal maps, Beltrami equation.

Theorem 2. *Let K_0 be a regular gauge in $\mathbf{R}^3$ and let K be any convex body in $\mathbf{R}^3$ such that for some constants α, β the equalities $w_K(u) = \alpha w_{K_0}(u)$ and $V_2(K|u^\perp) = \beta V_2(K_0|u^\perp)$ hold for all $u \in \mathbb{S}^{n-1}$. Then K is homothetic to K_0.*

Letting K_0 be a Euclidean ball recovers Theorem 1. While we are assuming some regularity on the gauge body K_0, the main point is that no assumptions, other than convexity, are being put on K. It is likely that the result also holds with no restrictions on either K or K_0. One indication this may be the case is a beautiful and surprising result of Schneider [20] that almost every, in the sense of Baire category, centrally convex body K_0 is determined up to translation in the class of all convex bodies by just its width function. This contrasts strongly with the fact that for any regular gauge K_0 there is an infinite dimensional family of convex bodies that have the same width function as K_0 (see Remark 2.7 below).

Two convex bodies K and K_0 in $\mathbf{R}^n$ have *proportional k-brightness* iff there is a constant γ such that $V_k(K|P) = \gamma V_k(K_0|P)$ for all k-dimensional subspaces P of $\mathbf{R}^n$. Theorem 2 implies a result, valid in all dimensions, about pairs of convex bodies that have both 1-brightness and 2-brightness proportional. If A and B are convex sets in $\mathbf{R}^n$ and L is a linear subspace of $\mathbf{R}^n$, then taking Minkowski sums commutes with projection onto L, that is $(A + B)|L = A|L + B|L$. As the projection of a Euclidean ball is a Euclidean ball, it follows that if K_0 is a regular gauge in $\mathbf{R}^n$, then $K_0|L$ is a regular gauge in L. Also, if P is a linear subspace of L, then $K|P = (K|L)|P$. Therefore if K_0 is a regular gauge in $\mathbf{R}^n$ and K is a convex body such that K and K_0 have proportional 1-brightness and proportional 2-brightness, then for any three dimensional subspace L of $\mathbf{R}^n$ the set $L|K_0$ is a regular gauge in L and $K_0|L$ and $K|L$ will have proportional 1-brightness and proportional 2-brightness as subsets of L. Thus by Theorem 2 $K|L$ is homothetic to $K_0|L$. However, if the projections $K_0|L$ and $K|L$ are homothetic for all three dimensional subspaces L, then, [7, Thm 3.1.3, p. 93], K is homothetic to K_0. Thus:

Corollary. *If K_0 is a regular gauge in $\mathbf{R}^n$, $n \geq 3$, and K is a convex body in $\mathbf{R}^n$ that has 1-brightness and 2-brightness proportional to those of K_0, then K is homothetic to K_0. In particular if K_0 a Euclidean ball this implies any convex body K in $\mathbf{R}^n$ of constant 1-brightness and 2-brightness is also a Euclidean ball.* $\square$

The contents of this the paper are as follows. In Section 2 some preliminaries about convex sets are given and a $C^{1,1}$ regularity result, Proposition 2.5, for the support functions of convex sets in $\mathbf{R}^n$ that appear is a summand in a convex set with $C^{1,1}$ support function is proven. (I am indebted to Daniel Hug for some of the results in this section). Section 3 gives explicit formulas, in terms of the support function, h, for the inverse of the Gauss map of the boundary of a convex set in $\mathbf{R}^n$ and conditions are given for two convex sets with $C^{1,1}$ boundary to have proportional brightness. It is important for our applications that some of these formulas (eg. Proposition 3.2) apply even when the function h is not the support function of a convex set. In Section 4 the results of the previous sections are applied to reduce the proof Theorem 2 to an analytic problem. In Section 5 the analytic result is proven by use of quasiconformal maps, the Beltrami equation, and the elementary theory of covering spaces.

2. Preliminaries on convexity.

We assume that $\mathbf{R}^n$ has its standard inner product $\langle\,,\rangle$ and let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbf{R}^n$. For any convex body K contained $\mathbf{R}^n$, the support function $h = h_K$ of K is the function $h\colon \mathbb{S}^{n-1} \to \mathbf{R}$ given by $h(u) := \max_{y\in K}\langle y, u\rangle$. A convex body is uniquely determined by its support function. The Minkowski sum of K_1 and K_2 corresponds to the sum of the support functions: $h_{K_1+K_2} = h_{K_1} + h_{K_2}$. The *width function* of K is $w = w_K$ is $w(u) = h(u) + h(-u)$. This is the length of the projection of K onto a line parallel to the vector u. In the terminology of Gardner, [7, p. 99], the *central symmetral* of a convex body K is the convex body $K_0 := \frac{1}{2}(K - K) = \{\frac{1}{2}(a - b) : a, b \in K\}$. The body K_0 is centrally symmetric about the origin, and, denoting the support function of K_0 by h_0, it follows from $h_{\frac{1}{2}(K-K)} = \frac{1}{2}h_K + \frac{1}{2}h_{-K}$ that $h_0(u) = \frac{1}{2}(h(u)+h(-u))$. Therefore K and K_0 have the same width in all directions. These definitions imply that a convex body has constant width w if and only if its central symmetral is a Euclidean ball of radius $w/2$.

We need the following, which is an elementary corollary of the Brunn-Minkowski theorem. For a proof see [7, Thm 3.2.2, p. 100].

2.1. **Proposition.** *The volumes of a convex body K and its central symmetral $K_0 = \frac{1}{2}(K - K)$ satisfy $V(K_0) \geq V(K)$ with equality if and only if K is a translate of K_0.* □

Recall that a function f defined on an open subset U of $\mathbf{R}^k$ is of class $C^{1,1}$ iff it is continuously differentiable and all the first partial derivatives satisfy a Lipschitz condition. A convex body K has $C^{1,1}$ boundary iff its boundary ∂K is locally the graph of a $C^{1,1}$ function.

There is a very nice geometric characterization of the convex bodies that have $C^{1,1}$ boundaries in terms of freely sliding bodies. Let K_1 and K_2 be convex bodies in $\mathbf{R}^n$. Then K_1 *slides freely inside of* K_2 iff for all $a \in \partial K_1$ there is a translate $y+K_2$ of K_2 such that $K_1 \subseteq y + K_2$ and $a \in y + K_2$. It is not hard to see, [19, Thm 3.2.2, p. 143], that K_1 slides freely inside of K_2 if and if K_1 is a Minkowski summand of K_2. That is, if and only if there is a convex set K such that $K + K_1 = K_2$. In what follows we will use the expressions "K_1 slides freely inside of K_2" and "K_1 is a Minkowski summand of K_2" interchangeably. A proof of the following can be found in [12, Prop. 1.4.3, p. 97].

2.2. **Proposition.** *A convex body K has $C^{1,1}$ boundary if and only if some Euclidean ball B_r slides freely inside of K.* □

I learned of the following elegant dual from of this theorem, with a somewhat different proof, from Daniel Hug.

2.3. **Proposition** (D. Hug [13])**.** *The support function h of a convex body K is $C^{1,1}$ if and only if K slides freely inside of some Euclidean ball B_R.*

Proof. Assume that K slides freely inside of the ball B_R of radius R. Without loss of generality it may be assumed that the origin is in the interior of K. Let $K^\circ := \{y : \langle y, x\rangle \leq 1 \text{ for all } x \in K\}$ be the *polar body* of K. The radial function of K° (which is the positive real valued function ρ on $\mathbb{S}^{n-1}$ such that $u \mapsto \rho(u)u$ parameterizes the boundary $\partial(K^\circ)$ of K°) is $\rho(u) = 1/h(u)$, [19, Rmk 1.7.7, p. 44]. So it is enough to show that ρ is a $C^{1,1}$ function, and to show this it is enough to

show that the boundary $\partial(K^\circ)$ is $C^{1,1}$. By Proposition 2.2 it is enough to show that some ball slides freely inside of K°. Let $\rho(u)u \in \partial(K^\circ)$. Becasue K slides freely inside a ball of radius R there is a ball $B(a, R)$ of radius centered at some point a such that $K \subset B(a, R)$ and a point $x \in K \cap \partial B(a, R)$ such that u is the outward pointing normal to $B(a, R)$ at x. As the operation of taking polars is inclusion reversing, $B_R(a)^\circ$ is contained in K° and as u is the outward pointing unit normal to both K and $B(a, R)$ at x we also have $\rho(u)u \in \partial(B_R(a)^\circ)$. The support function of $B_R(a)$ is $h_{B_R(a)}(u) = R + \langle a, u\rangle$ and therefore the radial function of the polar $B_R(a)^\circ$ is $\rho_{B_R(a)^\circ}(u) = 1/(R + \langle a, u\rangle)$. Thus points on $\partial(B_R(a)^\circ)$ are of the form $y = (1/(R + \langle a, u\rangle))\, u$ for $u \in \mathbb{S}^{n-1}$. This implies $|y| = 1/(R + \langle a, u\rangle)$ and $\langle a, y\rangle = \langle a, u\rangle/(R + \langle a, u\rangle)$. If $\langle a, u\rangle$ is ellimated from these equations the result can be written as

$$R^2|y|^2 - \langle a, y\rangle^2 + 2\langle a, y\rangle = 1.$$

For each a this is an ellipsoid and an ellipsoid has positive rolling radius (which is the largest number r so that a ball of radius r slides freely inside of the body). More generally for any ball $B_R(v)$ of radius R and center v containing K the polar $B_R(v)^\circ$ is an ellipsoid. By Blaschke's rolling theorem, [19, Cor. 3.2.10, p. 150], the rolling radius is the smallest radius of curvature of $\partial(B_R(v)^\circ)$ and this is a continuous function of the vector v. The set of v such that $B_R(v)$ contains K is a compact set and therefore, by the continuous dependence of the rolling radius of $\partial(B_R(v)^\circ)$ on v, there is a positive number r_0 such that a ball of radius r_0 slides freely inside of any $B_R(v)^\circ$ that contains K. In particular this is true of $B_R(a)^\circ$ and so K° contains an internally tangent ball of radius r_0 at $\rho(u)u$. But $\rho(u)u$ was an arbitrary point of $\partial(K^\circ)$ and whence a ball of radius r_0 slides freely inside of K° as required.

Conversely assume that the support function h of K is $C^{1,1}$. Let $\tilde{h}$ be the extension of h to $\mathbf{R}^n$ that is homogeneous of degree 1. Explicitly

$$\tilde{h}(x) := \max_{y \in K} \langle y, x\rangle. \tag{2.1}$$

As h is $C^{1,1}$ the function $\tilde{h}$ is $C^{1,1}_{\mathrm{Loc}}$ on $\mathbf{R}^n \setminus \{0\}$ and $\tilde{h}$ is convex, [19, Thm 1.7.1, p. 38], the distributional Hessian $\partial^2\tilde{h}$ will be positive semi-definite on $\mathbf{R}^n \setminus \{0\}$ and, because h is $C^{1,1}$, locally bounded above. Thus there is a positive real number R such that $H_0 := R\|\cdot\| - \tilde{h}$ is a convex function. But then, [19, Thm 1.7.1, p. 38], $H_0\big|_{\mathbb{S}^{n-1}}$ is the support function of a unique convex body K_0 and $H_0 + \tilde{h} = R\|\cdot\|$ implies that $K + K_0 = B_R(0)$. Therefore K is a summand in a ball. $\square$

2.4. **Corollary.** *Let K_0 be a body that is centrally symmetric about the origin, with ∂K_0 of class C^2 with positive Gauss curvature. Then K_0 is a regular gauge.*

Proof. It follows from a generalization Blaschke's rolling theorem, [19, Cor. 3.2.10, p. 150], that if B_r is a Euclidean ball with r smaller than any of the radii of curvature of K_0, that B_r slides freely inside of K_0 and if R is larger than any of the radii of curvature of ∂K_0, then K_0 slides freely inside of B_R. $\square$

2.5. **Proposition.** *Let $K_1, \dots, K_k$ be convex bodies in $\mathbf{R}^n$ such that the Minkowski sum $K_1 + \cdots + K_k$ has $C^{1,1}$ support function. Then each summand K_j also has $C^{1,1}$ support function.*

Proof. If $K_1 + \cdots + K_k$ has $C^{1,1}$ support function then, by Proposition 2.3, $K_1 + \cdots + K_k$ is a Minkowski summand in some ball B_R. But then each K_j is also

a summand in B_R and therefore Proposition 2.3 yields that K_j has $C^{1,1}$ support function. $\square$

2.6. **Corollary.** *Let K be a convex body such its central symmetral has a $C^{1,1}$ support function. Then the support function of K is also $C^{1,1}$. In particular any convex body of constant width has $C^{1,1}$ support function.*

Proof. If K_0 is the central symmetral of K, then $K+(-K)=2K_0$. As K_0 has $C^{1,1}$ support function, h_0, the support function, $2h_0$, of $2K_0$ is also $C^{1,1}$ and therefore the support function of K is $C^{1,1}$ by Proposition 2.5. $\square$

2.7. *Remark.* Corollary 2.6 is sharp in the sense that even when the support function, h_0, of the central symmetral, K_0, is C^∞, the most that can be said about the regularity of support function, h, of K is that it is $C^{1,1}$. For example let h_0 be the support function of a regular gauge, K_0, and let p a $C^{1,1}$ function $p\colon \mathbb{S}^{n-1}\to \mathbf{R}$ with $p(-u)=-p(u)$. Then for sufficiently small $\varepsilon>0$ the function $h:=h_0+\varepsilon p$ is the support function of a convex body with the same width function as K_0. But there are many choices of h_0 and p with h_0 of class C^∞ and h only of class $C^{1,1}$.

3. Support Functions and the Inverse of the Gauss Map.

We view vector fields ξ on subsets of U of $\mathbf{R}^n$ as functions $\xi\colon U\to\mathbf{R}^n$. A vector field on $\mathbb{S}^{n-1}$ is a function $\xi\colon\mathbb{S}^{n-1}\to\mathbf{R}^n$ such that for all $u\in\mathbb{S}^{n-1}$ the vector $\xi(u)\in T_u\mathbb{S}^{n-1}$. As the tangent space, $T_u\mathbb{S}^{n-1}$, to $\mathbb{S}^{n-1}$ at u is just $u^\perp$, the orthogonal compliment to u in $\mathbf{R}^n$, a vector field ξ on $\mathbb{S}^{n-1}$ can also be viewed as a map from $\mathbb{S}^{n-1}$ to $\mathbf{R}^n$ with $\xi(u)\perp u$ for all u. If $X\in T_u\mathbb{S}^{n-1}$ is a tangent vector to $\mathbb{S}^{n-1}$ at u, then a *curve fitting* X is a smooth curve $c\colon(a,b)\to\mathbb{S}^{n-1}$ defined on an interval about 0 with $c(0)=u$ and $c'(0)=X$. If ξ is a vector field on $\mathbb{S}^{n-1}$ that is differentiable at the point u, then for any $X\in T_u\mathbb{S}^{n-1}$ the *covariant derivative, $(\nabla_X\xi)(u)$, of ξ by X* is the projection of $\frac{d}{dt}\xi(c(t))\big|_{t=0}$ onto $T_u\mathbb{S}^{n-1}$ where c is any curve fitting X. This is independent of the choice of c fitting X and is given explicitly by

$$(\nabla_X\xi)(u) := \frac{d}{dt}\xi(c(t))\bigg|_{t=0} - \left\langle \frac{d}{dt}\xi(c(t))\bigg|_{t=0}, u\right\rangle u.$$

This definition implies that for any smooth curve $c\colon(a,b)\to\mathbb{S}^{n-1}$ and any vector field ξ on $\mathbb{S}^{n-1}$ that

$$\frac{d}{dt}\xi(c(t)) = (\nabla_X\xi)(c(t)) + \left\langle \frac{d}{dt}\xi(c(t)), c(t)\right\rangle c(t) \tag{3.1}$$

for any value t such that ξ is differentiable at $c(t)$.

For any C^1 function $p\colon\mathbb{S}^{n-1}\to\mathbf{R}$ the (spherical) *gradient* is the vector field, ∇p, on $\mathbb{S}^{n-1}$ such that $\langle\nabla p, X\rangle = dp(X)$ for all vectors X tangent to $\mathbb{S}^{n-1}$. At any point u where the vector field ∇p is differentiable the *second derivative* of p is the linear map $\nabla^2 p(u)\colon T_u\mathbb{S}^{n-1}\to T_u\mathbb{S}^{n-1}$ given by

$$\nabla^2 p(u)X := (\nabla_X\nabla p)(u).$$

3.1. *Remark.* There is a another way of viewing $\nabla^2 p$ that is useful. If p is defined on $\mathbb{S}^{n-1}$ then extend p to $\mathbf{R}^n$ to be homogeneous of degree one. That is let $\tilde p\colon\mathbf{R}^n\to\mathbf{R}$ be

$$\tilde p(x) = |x|p(|x|^{-1}x) \tag{3.2}$$

for $x \neq 0$ and $\tilde{p}(0) = 0$. Let $\partial\tilde{p}$ be the usual gradient of $\tilde{p}$, that is $\partial\tilde{p}$ is the column vector with components $\partial_1\tilde{p}, \partial_2\tilde{p}, \dots, \partial_n\tilde{p}$, and let $\partial^2\tilde{p}$ be the field of linear maps on $\mathbf{R}^n \smallsetminus \{0\}$ given by $\partial^2\tilde{p}(x)Y := (\partial_Y\partial\tilde{p})(x)$ where ∂_Y is the usual directional derivative in the direction of the vector Y. The matrix of $\partial^2\tilde{p}$ with respect to the coordinate basis is the usual Hessian matrix $[\partial_i\partial_j\tilde{p}]$. A strightforward calculation shows that $\partial^2\tilde{p}$ and $\nabla^2 p$ are related by

$$\partial^2\tilde{p}(x)Y = \frac{1}{|x|}\Big(\nabla^2 p(|x|^{-1}x) + p(|x|^{-1}x)I\Big)(Y - |x|^{-2}\langle Y, x\rangle x). \tag{3.3}$$

This implies that if $u \in \mathbb{S}^{n-1}$ and $Y \in T_u\mathbb{S}^{n-1} = u^\perp$, then

$$\partial^2\tilde{p}(u)Y = (\nabla^2 p(u) + p(u)I)Y$$

and $\partial^2\tilde{p}(u)u = 0$. Thus $T_u\mathbb{S}^{n-1}$ is invariant under $\partial^2\tilde{p}$. The symmetry of the second partials implies that when p is C^2, so that $\tilde{p}$ is C^2 on $\mathbf{R}^n \smallsetminus \{0\}$, then $\partial^2\tilde{p}(x)$ is self-adjoint (that is $\langle\partial^2\tilde{p}(x)X, Y\rangle = \langle X, \partial^2\tilde{p}(x)Y\rangle$) for $x \in \mathbf{R}^n \smallsetminus \{0\}$. But then $\nabla^2 p(u) = \partial^2\tilde{p}(u)\big|_{T_u\mathbb{S}^{n-1}} - p(u)I$ implies that $\nabla^2 p(u)$ is self-adjoint on $T_u\mathbb{S}^{n-1}$. The formula (3.3) also implies that $\nabla^2 p$ exists at $u \in \mathbb{S}^{n-1}$ if and only if $\partial^2\tilde{p}$ exists at all points tu with $t > 0$. This, combined with Fubini's Theorem, yields that $\nabla^2 p$ exists almost everywhere on $\mathbb{S}^{n-1}$ if and only if $\partial^2\tilde{p}$ exists almost everywhere on $\mathbf{R}^n$.

3.2. **Proposition.** *Let $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ be a Lipschitz map such that for all u where the derivative $\varphi'(u)$ exists it satisfies $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$. Then there is a unique $C^{1,1}$ function $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ such that*

$$\varphi(u) = p(u)u + \nabla p(u). \tag{3.4}$$

The derivative $\varphi'(u)$ exists at u if and only if the second derivative $\nabla^2 p(u)$ of p exists at u and at these points

$$\varphi'(u) = p(u)I + \nabla^2 p(u) \tag{3.5}$$

where I is the identity map on $T_u\mathbb{S}^{n-1}$. Conversely if p is $C^{1,1}$ and φ is given by 3.4 then $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$ for all points u where φ is differentiable. Finally for $k \geq 1$ the function φ is C^k if and only if p is C^{k+1}.

Proof. Any function $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ can be uniquely written as $\varphi(u) = p(u)u + \xi(u)$ where $p\colon \mathbb{S}^{n-1} \to \mathbf{R}$ and ξ is a vector field on $\mathbb{S}^{n-1}$. Because φ is Lipschitz, so are p and ξ. Therefore a theorem of Rademacher, [6, Thm 3.1.6, p. 216], implies that p and ξ are both differentiable almost everywhere on $\mathbb{S}^{n-1}$. Let E be the set of points where both p and ξ are differentiable. Then φ is also differentiable at u. Let $u \in E$, $X \in T_u\mathbb{S}^{n-1}$, and c a curve fitting X. Then, using (3.1),

$$\begin{aligned}
\varphi'(u)X &= \frac{d}{dt}\bigg|_{t=0} \big(p(c(t))c(t) + \xi(c(t))\big) \\
&= dp_u(X)u + p(u)X + (\nabla_X\xi)(u) + \left\langle \frac{d}{dt}\bigg|_{t=0} \xi(c(t)), u \right\rangle u.
\end{aligned}$$

But $dp_u(X) = \langle\nabla p(u), X\rangle$ and, using that $\langle\xi(c(t)), c(t)\rangle \equiv 0$,

$$\begin{aligned}
\left\langle \frac{d}{dt}\bigg|_{t=0} \xi(c(t)), u \right\rangle &= \frac{d}{dt}\bigg|_{t=0} \langle\xi(c(t)), c(t)\rangle - \langle\xi(c(t)), c'(t)\rangle\bigg|_{t=0} \\
&= -\langle\xi(u), X\rangle.
\end{aligned}$$

Therefore the formula for $\varphi'(u)X$ becomes

$$\varphi'(u)X = \langle \nabla p(u) - \xi(u), X\rangle u + p(u)X + (\nabla_X \xi)(u). \tag{3.6}$$

As $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ the component normal to $\mathbb{S}^{n-1}$ must vanish. Whence $\langle \nabla p(u) - \xi(u), X\rangle = 0$ for all $X \in T_u\mathbb{S}^{n-1}$. This implies

$$\xi(u) = \nabla p(u) \quad \text{at points } u \text{ where both } p \text{ and } \xi \text{ are differentiable.} \tag{3.7}$$

We now argue that p is continuously differentiable and that $\nabla p = \xi$ on all of $\mathbb{S}^{n-1}$. This will be based on the following elementary lemma, whose proof will be given after the proof of Proposition 3.2.

3.3. **Lemma.** *Let q be a real valued Lipschitz function defined on an open subset U of $\mathbf{R}^N$. Assume that there are Lipschitz functions $q_1, \dots, q_N$ on U and a set of full measure $S \subseteq U$ such that for all $x \in S$ the partial derivatives of q exist and satisfy $\partial_j q(x) = q_j(x)$ for all $x \in S$. Then q is of class $C^{1,1}$ and $\partial_j q = q_j$ on all of U.*

Near any point, u_0, of $\mathbb{S}^{n-1}$ there is a C^∞ parameterization $f\colon U \to V \subset \mathbb{S}^{n-1}$ of a neighborhood V of u_0, with U a bounded open set in $\mathbf{R}^{n-1}$, and f a C^∞ diffeomorphism. To show that p is $C^{1,1}$ it is enough to show the function $q\colon U \to \mathbf{R}$ given by $q(x) := p(f(x))$ is $C^{1,1}$. Let S be the subset of points $x \in U$ where both p and ξ are differentiable at $f(x)$. As p and ξ are Lipschitz and f is a diffeomorphism this is a set of full measure in U and at all points of S we have, by (3.7), that $\nabla p(f(x)) = \xi(f(x))$. As ξ is Lipschitz there are real valued Lipschitz functions $\xi^1, \dots, \xi^{n-1}$ defined on U such that $\xi(f(x)) = \sum_{i=1}^{n-1} \xi^i(x)\partial_i f(x)$. Therefore at points x in S we have $\nabla p(f(x)) = \xi(f(x)) = \sum_{i=1}^{n-1} \xi^i(x)\partial_i f(x)$ and thus

$$\partial_j q(x) = dp_{f(x)}(\partial_j f) = \langle \nabla p(f(x)), \partial_j f\rangle = \sum_{i=1}^{n-1} \xi^i(f(x))\langle \partial_i f(x), \partial_j f(x)\rangle.$$

The functions $q_j := \sum_{i=1}^{n-1} \xi^i(f(x))\langle \partial_i f(x), \partial_j f(x)\rangle$ are Lipschitz so Lemma 3.3 implies that q, and therefore also p, is a $C^{1,1}$ function and that ∇p is a Lipschitz.

By (3.7) $\nabla p(u) = \xi(u)$ on the dense set E and ∇p and ξ are continuous thus $\nabla p = \xi$ on all of $\mathbb{S}^{n-1}$. Therefore $\varphi(u)$ is given by (3.4) as required. When φ is of this form it is clear that φ is differentiable exactly at the points u where the second derivative $\nabla^2 p(u)$ exists. At such points use $\nabla p = \xi$ and $\nabla_X \xi(u) = (\nabla_X \nabla p)(u) = \nabla^2 p(u)$ in (3.6) to see that (3.5) holds. This completes the proof that if $\varphi\colon \mathbb{S}^{n-1} \to \mathbf{R}^n$ is a Lipschitz map with $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $u \in \mathbb{S}^{n-1}$ where φ is differentiable, then φ is given by (3.4) for a uniquely determined $C^{1,1}$ function p.

Conversely if p is $C^{1,1}$ let $\xi = \nabla p$ in the calculations leading up to (3.6) to see that φ given by (3.4) satisfies $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $u \in \mathbb{S}^{n-1}$ where φ is differentiable.

Finally $\varphi(u) = h(u)u + \nabla p(u)$ makes it clear that if h is C^{k+1}, then φ is C^k. Conversely if φ is C^k, then $p(u) = \langle u, \varphi(u)\rangle$ implies p is C^k. Then $\nabla p(u) = \varphi(u) - p(u)u$ implies that ∇p is also C^k. But if ∇p is C^k, then p is C^{k+1}. □

Proof of Lemma 3.3. We will show that the j-th distributional derivative of q is q_j. By definition this means we need to show that for all C^∞ functions ψ with compact support contained in U that $\int_U q\partial_j\psi\,dx = -\int_U q_j\psi\,dx$. Let e_j be the j-th

coordinate vector. Then

$$\begin{aligned}
\int_U q(x)\partial_j\psi(x)\,dx &= \lim_{h\to 0}\int_U q(x)\frac{\psi(x+he_j)-\psi(x)}{h}\,dx \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_U q(x)\psi(x+he_j)\,dx - \frac{1}{h}\int_U q(x)\psi(x)\,dx\right) \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_U q(x-he_j)\psi(x)\,dx - \frac{1}{h}\int_U q(x)\psi(x)\,dx\right) \\
&= \lim_{h\to 0}\int_U \frac{q(x-he_j)-q(x)}{h}\psi(x)\,dx.
\end{aligned}$$

But q is Lipschitz and therefore the quotients $(q(x-he_j)-q(x))/h$ are uniformly bounded. By assumption for all $x\in S$, $\lim_{h\to 0}(q(x-he_j)-q(x))/h = -\partial_j q(x) = -q_j(x)$ and S has full measure so this limit holds almost everywhere. Therefore Lebesgue's bounded convergence theorem implies $\lim_{h\to 0}\int_U((q(x-he_j)-q(x))/h)\psi(x))\,dx = -\int_U q_j q(x)\psi(x)\,dx$. Using this in the calculation above yields that$\int_U q\partial_j\psi\,dx = -\int_U q_j\psi\,dx$ holds, and thus the distributional partial derivatives $\partial_j q$ are q_j. Then a standard result about distributional derivatives, [11, Thm 1.4.2, p. 10], implies that the classical partial derivatives $\partial_j q$ of q are equal to q_j in all of U. But a function with continuous partial derivatives is C^1. Finally $\partial_j q = q_j$ so the derivative is Lipschitz, that is q is of class $C^{1,1}$. □

3.4. **Proposition.** *Let $p\colon \mathbb{S}^{n-1}\to \mathbf{R}$ be a $C^{1,1}$ function. Then for almost all $u\in\mathbb{S}^{n-1}$ the second derivative $\nabla^2 p(u)$ exists and is self-adjoint.*

Proof. If p is $C^{1,1}$ the vector field ∇p is Lipschitz and thus by Rademacher's Theorem $\nabla^2 p(u)$ exists for almost all u. We have seen, Remark 3.1, that if p is of class C^2, then $\nabla^2 p(u)$ is self-adjoint for all $u\in\mathbb{S}^{n-1}$. In the case that p is $C^{1,1}$, for each $\varepsilon>0$ there is a C^2 function p_ε such that if $E_\varepsilon := \{u\in\mathbb{S}^{n-1} : p(u) = p_\varepsilon(u), \nabla p(u) = \nabla p_\varepsilon(u), \nabla^2 p(u) = \nabla^2 p_\varepsilon(u)\}$ then the measure of $\mathbb{S}^{n-1}\smallsetminus E_\varepsilon$ is less than ε, [6, Thm 3.1.15, p. 227]. As p_ε is C^2, $\nabla^2 p(u) = \nabla^2 p_\varepsilon(u)$ is self-adjoint for all $u\in E_\varepsilon$. Letting ε go to zero shows that $\nabla^2 p$ is self-adjoint almost everywhere on $\mathbb{S}^{n-1}$. □

Before applying Proposition 3.2 to the support function of a convex set, it is useful to record some symmetry properties of the operators ∇ and ∇^2. Note that the tangent spaces $T_u\mathbb{S}^{n-1}$ and $T_{-u}\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$ at antipodal points u and $-u$ are both just the orthogonal compliment $u^\perp$ to u. Therefore for a function p on $\mathbb{S}^{n-1}$ the vectors $\nabla p(u)$ and $\nabla p(-u)$ are in the same vector space, $u^\perp$, and the linear maps $\nabla^2 p(u)$ and $\nabla^2 p(-u)$ act on the same vector space $u^\perp$. Recall that a function $p\colon\mathbb{S}^{n-1}\to\mathbf{R}$ is even (respectively odd) iff $p(-u)=p(u)$ (respectively $p(-u) = -p(u)$). These definitions extend in a obvious way to vector fields or fields of linear maps on $\mathbb{S}^{n-1}$. The proof of the following is elementary and left to the reader.

3.5. **Lemma.** *Let $p\colon\mathbb{S}^{n-1}\to\mathbf{R}$ be a $C^{1,1}$ function. If p is even, then ∇p is odd, and $\nabla^2 p$ is even. If p is odd, then ∇p is even, and $\nabla^2 p$ is odd. (As p is $C^{1,1}$ the tensor $\nabla^2 p$ will only be defined almost everywhere. Saying this it is even (or odd) means that $\nabla^2 p(u)$ is defined if and only if $\nabla^2(-u)$ is defined and at these points $\nabla^2 p(-u) = \nabla^2 p(u)$ (or $\nabla^2 p(-u) = -\nabla^2 p(u)$).)* □

Recall that if K is a convex body with C^1 boundary ∂K, then the *Gauss map* is the function $\nu\colon \partial K \to \mathbb{S}^{n-1}$ where $\nu(x) = u$ iff u is the (unique as ∂K is C^1) outward pointing unit vector to K at x. If h is the support function of K, then it is not hard to see that $h(\nu(x)) = \langle x, \nu(x)\rangle$, Therefore, if ν is injective, so that ν^{-1} exists, then $h(u) = \langle \nu^{-1}(u), u\rangle$, [19, p. 106]. More generally when the support function h is C^1 the function $\varphi(u) = h(u)u + \nabla h(u)$ can still be viewed as the inverse of the Gauss map:

3.6. **Proposition.** *Let K be a convex body in $\mathbf{R}^n$ with C^1 support function h. Then the map $\varphi(u) = h(u)u+\nabla h(u)$ maps $\mathbb{S}^{n-1}$ onto ∂K with the property that $\varphi(x) = u$ if and only if u is an outward unit normal to K at x.*

Proof. We first assume that ∂K is C^∞ with positive curvature. Then the Gauss map $\nu\colon \partial K \to \mathbb{S}^{n-1}$ is a diffeomorphism. Let $\varphi := \nu^{-1}\colon \mathbb{S}^{n-1} \to \partial K$ be the inverse of ν. Then φ is a diffeomorphism and $T_u\mathbb{S}^{n-1}$ and $T_{\varphi(u)}\partial K$ are the same (as we are identifying subspaces that differ by a parallel translation). Whence $\varphi'(u)X \in T_u\mathbb{S}^{n-1}$ for all $X \in T_u\mathbb{S}^{n-1}$. By Proposition 3.6 this implies there is a unique smooth real valued function p on $\mathbb{S}^{n-1}$ such that $\varphi(u) = p(u)u+\nabla p(u)$. Then $p(u) = \langle \varphi(u), u\rangle$. But, from the remarks above, the support function of K is also given by $h(u) = \langle \varphi(u), u\rangle$ and therefore $p = h$. So in this case $\varphi(u) = h(u)u+\nabla h(u)$ is the inverse of the Gauss map and so $\varphi(u) = x$ if and only if u is the outward normal to K at x is clear.

Now assume that h is C^1 and set $\varphi(u) = h(u)u+\nabla h(u)$. Then φ is a continuous map from $\mathbb{S}^{n-1}$ to $\mathbf{R}^n$. There are convex bodies $\{K_\ell\}_{\ell=1}^\infty$ whose boundaries are smooth with positive curvature and such that if the support function of K_ℓ is h_ℓ, then $h_\ell \to h$ in the C^1 topology, [19, pp. 158–160]. Therefore if $\varphi_\ell(u) := h_\ell(u)u + \nabla h_\ell(u)$, then $\varphi_\ell \to \varphi$ uniformly. The Hausdorff distance (see [19, p. 48] for the definition) between K and K_ℓ is given in terms of the support functions by $d_{\mathrm{Hau}}(K, K_\ell) = \|h - h_\ell\|_{L^\infty}$, [19, 1.8.11, p. 53], and so $K_\ell \to K$ in the Hausdorff metric. Because K and K_ℓ are convex this implies $\partial K_\ell \to \partial K$ in the Hausdorff metric. As $\varphi_\ell(u) \in \partial K_\ell$ this yields $\varphi(u) = \lim_{\ell\to\infty} \varphi_\ell(u) \in \partial K$. Therefore φ maps $\mathbb{S}^{n-1}$ into ∂K. Let $x \in \partial K$ and let u be an outward pointing unit normal to K at x. Then u is an outward pointing normal to K_ℓ at $\varphi_\ell(u)$. Therefore the half space $H_\ell^- := \{y \in \mathbf{R}^n : \langle y, u\rangle \le h_\ell(u)\}$ contains K_ℓ and its boundary ∂H_ℓ^- is a supporting hyperplane to K_ℓ at $\varphi_\ell(u)$. Using that $h_\ell \to h$ uniformly, that $K_\ell \to K$ in the Hausdorff metric, and that $\varphi_\ell(u) \to \varphi(u)$ we see that K is contained in $H^+ := \{y \in \mathbf{R}^n : \langle y, u\rangle \le h(u)\}$ and that $x \in \partial H^+$. Thus u is an outward pointing unit normal to K at $\varphi(u)$. But, [19, Cor. 1.7.3, p. 40], if the support function is differentiable, then the body is strictly convex. Therefore K is strictly convex and thus a unit vector can be an outward unit normal to K in at most one point. So, as u is an outward unit normal to K at $\varphi(u)$ and at x, we have $\varphi(u) = x$.

Summarizing, if $x \in \partial K$ and u is an outward unit normal to K at x, then $\varphi(u) = x$. But for any point of ∂K there is at least one unit normal u to K at x, so $\varphi\colon \mathbb{S}^{n-1} \to \partial K$ is surjective. To finish we need that if $\varphi(u) = x$, then u is an outward pointing unit normal to K at x. The vector u will be an outward pointing unit normal to K at some point $y \in \partial K$. But then $\varphi(u) = y$. Thus $x = y$ and u is an outward pointing unit vector to K at x. $\square$

3.7. **Proposition.** *Let K be a compact body with $C^{1,1}$ support function h. Then $hI + \nabla^2 h$ is positive semi-definite almost everywhere on $\mathbb{S}^{n-1}$. If in addition there*

is a Euclidean ball that slides freely inside of $\mathbb{S}^{n-1}$*, then there is a positive constant* C_1 *such that* $\det(hI + \nabla^2 h) \ge C_1$ *almost everywhere on* $\mathbb{S}^{n-1}$*.*

Proof. Let $\tilde{h}$ be the extension of h to $\mathbf{R}^n$ as a homogeneous function of degree one (thus $\tilde{h}$ is given by both the formulas (2.1) and (3.2)). The function $\tilde{h}$ is convex, [19, Thm 1.7.1, p. 38], and therefore its Hessian $\partial^2\tilde{h}$ is positive semi-definite at all points where it exists and is self-adjoint. But then the formula (3.3) relating $\partial^2\tilde{h}$ and $\nabla^2 h$ together with Remark 3.1 and Proposition 3.4, shows that $hI + \nabla^2 h$ is positive semi-definite almost everywhere on $\mathbb{S}^{n-1}$.

Assume that the Euclidean ball B_{2r} of radius $2r$ slides freely inside of K. Then there is a convex set K_1 such that $K_1 + B_{2r} = K$. However K_1 may not be a convex body. But $K_1 + B_{2r} = (K_1 + B_r) + B_r$ and $K_1 + B_r$ is a convex body. So by replacing K_1 by $K_1 + B_r$ we can assume $K_1 + B_r = K$ with K_1 a convex body. Let h_1 be the support function of K_1. Then, as the support function of B_r is the constant r, $h_1 + r = h$. This implies that h_1 is also $C^{1,1}$ and therefore $(h_1 I + \nabla^2 h_1)$ is positive semi-definite almost everywhere. But for any positive semi-definite matrices A and B the inequality $\det(A + B) \ge \det(A)$ holds. Therefore

$$\det(hI + \nabla^2 h) = \det(rI + (h_1 I + \nabla^2 h_1)) \ge \det(rI) = r^{n-1} =: C_1.$$

almost everywhere. □

3.8. **Lemma.** *Let* K *be a convex body in* $\mathbf{R}^n$ *with* $C^{1,1}$ *support function* h*. Then for any unit vector* $a \in \mathbf{R}^n$*,*

$$2V_{n-1}(K|a^\perp) = \int_{\mathbb{S}^{n-1}} \det(hI + \nabla^2 h)|\langle a, u\rangle|\, dV_{n-1}(u).$$

Proof. Let h be the support function of K and let $\varphi\colon \mathbb{S}^{n-1} \to \partial K$ be $\varphi(u) = h(u)u + \nabla h(u)$. By Proposition 3.6 φ maps $\mathbb{S}^{n-1}$ onto ∂K and, as h is $C^{1,1}$, the map φ is Lipschitz. As φ is Lipschitz it is differentiable almost everywhere and by Proposition 3.2 at the points u where it is differentiable $\varphi'(u) = h(u)I + \nabla^2 h(u)$. Let $f\colon \mathbb{S}^{n-1} \to K|a^\perp$ be the function $f(u) = \varphi(u)|a^\perp$. This maps $\mathbb{S}^{n-1}$ onto $K|a^\perp$. An elementary computation shows that the Jacobian, $J(f)(u) := \det(f'(u))$, of f is given by $J(f)(u) = \det\big(h(u)I + \nabla^2 h(u)\big)\,\langle a, u\rangle$. The area theorem, [6, Thm. 3.2.3, p. 243], (note that the definition of Jacobian used in the area theorem is the absolute value of the one being used here) implies

$$\begin{aligned}\int_{K|a^\perp} \#(f^{-1}[y])\, dV_{n-1}(y) &= \int_{\mathbb{S}^{n-1}} |J(f)(u)|\, dV_{n-1}(u)\\ &= \int_{\mathbb{S}^{n-1}} \det\big(h(u)I + \nabla^2 h(u)\big)\, |\langle a, u\rangle|\, dV_{n-1}(u)\end{aligned}$$

where $\#(f^{-1}[y])$ is the number of points in the preimage $f^{-1}[y] := \{x : f(x) = y\}$. To complete the proof it is enough to show $\#(f^{-1}[y]) = 2$ for almost all $y \in K|a^\perp$.

As $K|a^\perp$ is convex its boundary $\partial(K|a^\perp)$ has measure zero. Therefore we only need consider y in the interior, $\operatorname{int}\big(K|a^\perp\big)$, of $K|a^\perp$. If $y \in \operatorname{int}\big(K|a^\perp\big)$ then there are exactly two points $x_1, x_2 \in \partial K$ with $x_j|a^\perp = y$. Thus $f^{-1}[y]$ is the disjoint union of $\varphi^{-1}[x_1]$ and $\varphi^{-1}[x_2]$. But, [19, Thm 2.2.4, p. 74], the set, P, of points x in ∂K such that there is more than one outward unit normal to K at x is a set of measure zero. So if $x_1, x_2 \notin P$, each of the sets $\varphi^{-1}[x_1]$ and $\varphi^{-1}[x_2]$ will have just one elment and therefore $\#(f^{-1}[y]) = 2$. The map $y \mapsto y|a^\perp$ is Lipschitz and therefore it maps sets of measure zero to sets of measure zero. Thus $P|a^\perp$ is a set

of measure zero. Whence for $y \in \operatorname{int}\big(K|a^\perp\big) \setminus P|a^\perp$, and therefore for almost all $y \in K|a^\perp$, $\#(f^{-1}[y]) = 2$ which finishes the proof. □

3.9**. Proposition.** *Let K_1 and K_2 be convex bodies in $\mathbf{R}^n$ with $C^{1,1}$ support functions h_1 and h_2 respectively. Then there is a constant β such that $V_{n-1}(K_1|a^\perp) = \beta V_{n-1}(K_2|a^\perp)$ for all $a \in \mathbb{S}^{n-1}$ if and only if*

$$\det(h_1 I + \nabla^2 h_1) = \beta \det(h_2 I + \nabla^2 h_2) + q, \quad \text{with } q \text{ an odd function.}$$

Proof. By Lemma 3.8 $V_{n-1}(K_1|a^\perp) = \beta V_{n-1}(K_2|a^\perp)$ for all $a \in \mathbb{S}^{n-1}$ if and only if $\int_{\mathbb{S}^{n-1}} q(u)|\langle a,u\rangle|\,du = 0$ for all $a \in \mathbb{S}^{n-1}$ where $q = \det(h_1 I + \nabla^2 h_1) - \beta\det(h_2 I + \nabla^2 h_2)$. That is, if and only if q is in the kernel of the cosine transform $(Cf)(a) := \int_{\mathbb{S}^{n-1}} f(u)|\langle a,u\rangle|\,du$. But, [7, Thm C,2.4, p. 381], the kernel of the cosine transform is exactly the set of odd functions on $\mathbb{S}^{n-1}$. □

4. Three dimensional Bodies of Constant Width and Brightness.

To prove Theorem 2 we let K and K_0 be convex bodies in $\mathbf{R}^3$ such that K_0 is centrally symmetric about the origin and that there are constants α and β such that $w_K(u) = \alpha w_{K_0}(u)$ and $(K|u^\perp) = \beta V_2(K_0|y^\perp)$ for all unit vectors u. By rescaling K by a factor of $1/\alpha$ we can assume that $\alpha = 1$, that is K and K_0 have same width in all directions. Then K_0 being centrally symmetric about the origin implies that K_0 is the central symmetral $\frac{1}{2}(K - K)$ of K. Therefore to prove Theorems 1 and 2 it is enough to prove:

4.1**. Theorem.** *Let K be a convex body in $\mathbf{R}^3$ such that its central symmetral $K_0 = \frac{1}{2}(K-K)$ is a regular gauge and for some constant β*

$$V_2(K|u^\perp) = \beta V_2(K_0|u^\perp) \quad \text{for all } u \in \mathbb{S}^2 \tag{4.1}$$

Then K is a translate of K_0.

4.2**. Lemma.** *If* (4.1) *holds, then $\beta \le 1$ and if $\beta = 1$, then K is a translate of K_0.*

Proof. Let $u \in \mathbb{S}^2$. Then $K_0|u^\perp$ is centrally symmetric about the origin and, viewed as convex bodies in the two dimensional space $u^\perp$, the sets $K_0|u^\perp$ and $K|u^\perp$ have the same width function. Therefore $K_0|u^\perp$ is the central symmetral of $K|u^\perp$. By Proposition 2.1 this implies $V_2(K_0|u^\perp) \ge V_2(K|u^\perp)$ with equality if and only if $K|u^\perp$ is a translate of $K_0|u^\perp$. As $V_2(K|u^\perp) = \beta V_2(K_0|u^\perp)$ this yields that $\beta \le 1$. If $\beta = 1$, then for all $u \in \mathbb{S}^2$ the set $K|u^\perp$ is a translate of $K_0|u^\perp$. This implies, [7, Thm 3.1.3, p. 93], that K is a translate of K_0. □

From now on we assume K and K_0 satisfy the hypothesis of Theorem 4.1 and that h and h_0 are the support functions of K and K_0 respectively. By Lemma 4.2 if $\beta = 1$, Theorem 4.1 holds, so, towards a contradiction, assume $\beta < 1$.

4.3**. Lemma.** *If $\beta < 1$ then h and h_0 are related by $h = h_0 + p$ where p is an odd function. The function p satisfies*

1. *p is of class $C^{1,1}$,*
2. *The equality*

$$\det(pI + \nabla^2 p) = -(1-\beta)\det(h_0 I + \nabla^2 h_0) \tag{4.2}$$

holds almost everywhere on $\mathbb{S}^{n-1}$. Therefore there is a constant $\delta_0 > 0$ such that

$$\det(pI + \nabla^2 p) \le -\delta_0 \tag{4.3}$$

almost everywhere on $\mathbb{S}^{n-1}$*.*

(3) *If* $\varphi\colon \mathbb{S}^2 \to \mathbf{R}^3$ *is given by* $\varphi(u) = p(u)u + \nabla p(u)$ *then* φ *is Lipschitz and* $\varphi(-u) = \varphi(u)$*.*

Proof. As K and K_0 have the same width function, $h(u) + h(-u) = h_0(u) + h_0(-u) = 2h_0(u)$ as $h_0(-u) = h_0(u)$ because K_0 is centrally symmetric about the origin. Therefore

$$h(u) = \frac{1}{2}(h(u) + h(-u)) + \frac{1}{2}(h(u) - h(-u)) = h_0(u) + p(u)$$

where $p(u) := \frac{1}{2}(h(u) - h(-u))$ is clearly an odd function.

As K_0 is a regular gauge it slides freely inside of some Euclidean ball and thus by Proposition 2.3 h_0 is $C^{1,1}$. Then Corollary 2.6 implies h is $C^{1,1}$ and the formula $p(u) = \frac{1}{2}(h(u) - h(-u))$ shows that p is also $C^{1,1}$.

Proposition 3.9 implies there is an odd function q on $\mathbb{S}^2$ such that

$$\det(hI + \nabla^2 h) = \beta \det(h_0 I + \nabla^2 h_0) + q \tag{4.4}$$

holds almost everywhere on $\mathbb{S}^2$. The equality $h = h_0 + p$ implies

$$\det(hI + \nabla h) = \det\big((pI + \nabla^2 p) + (h_0 I + \nabla^2 h_0)\big)\,. \tag{4.5}$$

For any 2×2 matrix $\operatorname{tr}(A)^2 - \operatorname{tr}(A^2) = 2\det(A)$, where $\operatorname{tr}(A)$ is the trace of A. Define $\sigma(A,B)$ on pairs of 2×2 matrices by $\sigma(A,B) = \frac{1}{2}(\operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB))$. Then $\sigma(\,,\,)$ is a symmetric bilinear form and $\sigma(A,A) = \det(A)$. Whence $\det(A+B) = \det(A) + 2\sigma(A,B) + \det(B)$. Using this in (4.5) gives

$$\det(hI+\nabla^2 h) = \det(pI+\nabla^2 p)+2\sigma(pI+\nabla^2 p, h_0 I+\nabla^2 h_0)+\det(h_0 I+\nabla^2 h_0). \tag{4.6}$$

The function h_0 is even on $\mathbb{S}^2$ and Lemma 3.5 implies $\nabla^2 h_0$ is also even. Therefore $h_0 I + \nabla^2 h_0$ is even. Likewise Lemma 3.5 applied to the odd function p implies $pI + \nabla^2 p$ is odd. But $\det(-A) = \det(A)$ for 2×2 matrices, so the function $\det(pI + \nabla^2 p)$ is even. The function $\sigma(pI + \nabla^2 p, h_0 I + \nabla^2 h_0)$ is odd as a function of the first argument and even as a function of the second argument, therefore $\sigma(pI + \nabla^2 p, h_0 I + \nabla^2 h_0)$ is an odd function. Comparing the two formulas (4.4) and (4.6) for $\det(hI + \nabla^2 h)$ and equating the even parts gives

$$\beta \det(h_0 I + \nabla^2 h_0) = \det(pI + \nabla^2 p) + \det(h_0 I + \nabla^2 h_0).$$

This implies (4.2). By Proposition 3.7 and the assumption that K_0 slides freely inside of a Euclidean ball there is a constant $C_1 > 0$ such that $\det(h_0 I+\nabla^2 h_0) \geq C_1$. Then (4.2) implies (4.3) holds with $\delta_0 = (1-\beta)C_1$.

That p is $C^{1,1}$ implies $\varphi(u) = p(u)u + \nabla p(u)$ is Lipschitz. The function p is odd and, by Lemma 3.5, the vector field ∇p is even. Therefore $\varphi(-u) = p(-u)(-u) + \nabla p(-u) = p(u)u + \nabla p(u) = \varphi(u)$. $\square$

Letting p and $\varphi(u) = p(u)u + \nabla p(u)$ be as in the last lemma, for any unit vector a let $H_a := \langle \varphi(x), a\rangle$ be the height function of φ in the direction a. The following, which is trivial when h is C^2 (so that φ is C^1), is the main geometric fact behind the proof of Theorem 4.1.

4.4**. Claim.** *If the height function* H_a *has a local maximum or minimum at* u_0*, then* $u_0 = \pm a$*.*

Proof of Theorem 4.1 assuming the Claim. By compactness of $\mathbb{S}^2$ and the continuity of the height function H_a, there are points $u_1, u_2 \in \mathbb{S}^2$ such that $H_a(u_1)$ is a global minimum and $H_a(u_2)$ is a global maximum of H_a. By the claim $u_1 = \pm a$ and $u_2 = \pm a$, and therefore $u_1 = \pm u_2$. By Lemma 4.3, φ is an even function on $\mathbb{S}^2$ and whence

$$H_a(u_1) = \langle \varphi(u_1), a\rangle = \langle \varphi(\pm u_2), a\rangle = \langle \varphi(u_2), a\rangle = H_a(u_2).$$

As $H_a(u_1)$ and $H_a(u_2)$ are the minimum and maximum of H_a this implies $H_a(u)$ is constant. But this is true for any choice of a, so φ is constant. Then $\varphi'(u) = 0$ for all $u \in \mathbb{S}^2$. However, by Proposition 3.2, $\varphi'(u) = p(u)I + \nabla^2 p(u)$ for almost all $u \in \mathbb{S}^2$ and, by Lemma 4.3, $\det(pI + \nabla^2 p) < 0$ almost everywhere, which implies $\varphi'(u) \neq 0$ for almost all u. This contradiction completes the proof. □

We now reduce the claim to an analytic lemma that is proven in the next section. Let e_1, e_2, e_3 be the standard basis of $\mathbf{R}^3$. By a rotation we can assume that the height function, H_a, has a local maximum or maximum at e_3. Then to prove the claim we need to show that $a = \pm e_3$. We parameterize the open upper hemisphere $\mathbb{S}^2_+$ of $\mathbb{S}^2$ by

$$u = u(x, y) := \begin{bmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{bmatrix} \tag{4.7}$$

where $(x, y) \in \Delta_1 := \{(x, y) : x^2 + y^2 < 1\}$. The function p restricted to $\mathbb{S}^2_+$ can be expressed in terms of the coordinates x, y. Direct calculation shows

$$\nabla p = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}, u \right\rangle u = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} - (xp_x + yp_y) \begin{bmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{bmatrix}$$

Therefore φ is given by

$$\varphi(x, y) = pu + \nabla p = \begin{bmatrix} xp \\ yp \\ p\sqrt{1 - x^2 - y^2} \end{bmatrix} + \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} - (xp_x + yp_y) \begin{bmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{bmatrix}$$

and

$$\langle \varphi, e_3 \rangle = (p - (xp_x + yp_y))\sqrt{1 - x^2 - y^2}.$$

As p is of class $C^{1,1}$, Taylor's theorem implies $p(x, y) = p(0, 0) + xp_x(0, 0) + yp_y(0, 0) + O(x^2 + y^2)$, $xp_x(x, y) = xp_x(0, 0) + O(x^2 + y^2)$, and $yp_y(x, y) = yp_y(0, 0) + O(x^2 + y^2)$. Therefore

$$\langle \varphi, e_3 \rangle = p(0, 0) + O(x^2 + y^2). \tag{4.8}$$

We also consider the projection of φ onto the first two coordinates:

$$\psi(x, y) := \begin{bmatrix} p_x + x(p - xp_x - yp_y) \\ p_y + y(p - xp_x - yp_y) \end{bmatrix}.$$

This is clearly Lipschitz in a neighborhood of the origin.

4.5. **Main Lemma.** *With ψ as above, there is an open neighborhood W of $\psi(0, 0)$ in $\mathbf{R}^2$ and a constant C_o such that for all $w \in W$ there is a $z \in \Delta_1$ with $\psi(z) = w$ and $C_o^{-1}|z| \leq |w - \psi(0, 0)| \leq C_o|z|$.*

Assuming this we prove Claim 4.4. Write the unit vector a defining the height function H_a as $a = \tilde{a} + a_3 e_3$ where $\tilde{a} \in \mathbf{R}^2$ and $a_3 \in \mathbf{R}$. Then for $z = (x, y) \in \Delta_1$ and using (4.8)

$$\begin{aligned} H_a(z) &= \langle \varphi(z), a\rangle = \langle \psi(z), \tilde{a}\rangle + a_3 \langle \varphi(z), e_3\rangle \\ &= \langle \psi(z), \tilde{a}\rangle + a_3 p(0,0) + O(|z|^2). \end{aligned} \tag{4.9}$$

For real t with $|t|$ small let $w_t = \psi(0,0) + t\tilde{a}$. By Lemma 4.5 there is a $z_t \in \Delta_1$ with $\psi(z_t) = w_t$ and

$$|z_t| \leq C_o |w_t - \psi(0,0)| = C_o |\tilde{a}||t|.$$

Thus $|z_t|^2 = O(t^2)$. Using this in (4.9) gives

$$\begin{aligned} H_a(z_t) &= \langle \psi(0,0) + t\tilde{a}, \tilde{a}\rangle + a_3 p(0,0) + O(|z_t|^2) \\ &= (\langle \psi(0,0), \tilde{a}\rangle + a_3 p(0,0)) + t|\tilde{a}|^2 + O(t^2) \end{aligned}$$

This can only have a local maximum or minimum at $t = 0$ if $\tilde{a} = 0$. As a is a unit vector this implies that $a = \pm e_3$ and completes the proof of Claim 4.4.

5. Quasiconformal maps and the proof of the main lemma.

5.1. **Preliminaries on quasiconformal maps and the Beltrami equation.** We recall some basic definitions and facts about quasiconformal maps. We identify the complex numbers $\mathbf{C}$ with the real plane $\mathbf{R}^2$. Let $U \subseteq \mathbf{C}$ be an open set. If $f \colon U \to \mathbf{C}$ write $f = u + iv$. The function f is in the Sobolev space $W^{1,2}_{\mathrm{Loc}}(U)$ iff its distributional first derivatives are measurable functions that are square integrable on any compact subset of U. If $f \in W^{1,2}_{\mathrm{Loc}}(U)$, then the partial derivatives $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$ exist almost everywhere on U. If $f \colon U \to V$ is a homeomorphism between open sets U and V and also $f \in W^{1,2}_{\mathrm{Loc}}(U)$ a theorem of Gehring and Lehto [9] (cf. [1, Lem. 1, p. 24]) implies that f is differentiable almost everywhere (where the derivative, $f'(z)$, is a real linear map $f'(z) \colon \mathbf{R}^2 \to \mathbf{R}^2$). The *operator norm* of the linear map $f'(z)$ is $\|f'(z)\| := \sup_{|v|=1} |f'(z)v|$ and the Jacobian is $J(f)(z) = \det(f'(z)) = u_x v_y - u_y v_x$. For $K \geq 1$ a homeomorphism $f \colon U \to V$ between two open subsets of $\mathbf{C}$ is K-*quasiconformal* iff $f \in W^{1,2}_{\mathrm{Loc}}(U)$ and

$$\|f'(z)\|^2 \leq K J(f)(z)$$

holds almost everywhere in U. There are other equivalent analytic definitions of K-quasiconformality (cf. [1, p. 24], [15, pp. 6–7], [14, p. 5]). Introducing the complex derivatives $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$ and $\partial/\partial \overline{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ an equivalent definition for a homeomorphism $f \colon U \to V$ which is in $W^{1,2}_{\mathrm{Loc}}(U)$ to be K-quasiconformal is that

$$|f_{\overline{z}}| \leq \frac{K-1}{K+1} |f_z|$$

almost everywhere on U. There is a geometric definition of K-quasiconformal (for example see [1, p. 21]) that has the advantage that it makes it clear that a homeomorphism $f \colon U \to V$ between open subsets of $\mathbf{C}$ is K-quasiconformal if and only if its inverse $f^{-1} \colon V \to U$ is K-quasiconformal. The equivalence of the geometric and analytic definitions of K-quasiconformal was proven by Gehring and Lehto in [9] (cf. [1, Chap. II]). A corollary of the Gehring and Lehto theorem is the following (which can also be found explicitly in [15, Thm 4, p. 9]).

5.1. **Proposition.** *If $f\colon U \to V$ is a K-quasiconformal map between open subsets of $\mathbf{C}$, then the inverse $f^{-1}\colon V \to U$ is also K-quasiconformal and satisfies*

$$(f^{-1})_w = \frac{\overline{f_z}}{|f_z|^2 - |f_{\overline{z}}|^2}, \qquad (f^{-1})_{\overline{w}} = \frac{-f_{\overline{z}}}{|f_z|^2 - |f_{\overline{z}}|^2}.$$

almost everywhere on V. □

This implies a result on the Lipschitz invertiblity of certain homeomorphisms. Let $A \geq 1$, then an open connected subset V of $\mathbf{C}$ has *A-uniformly bounded intrinsic distances* iff any two points $w_0, w_1 \in V$ can be joined by a smooth curve c contained in V with $\mathrm{Length}(c) \leq A|w_1 - w_0|$.

5.2. **Proposition.** *Let $f\colon U \to V$ be a homeomorphism between open connected subsets of $\mathbf{C}$ such that the distributional first derivatives of f are bounded measurable functions and such that the Jacobian satisfies $J(f) \geq \delta$ almost everywhere for some positive constant δ. Also assume V has A-uniformly bounded intrinsic distances for some $A \geq 1$. Then the inverse $f^{-1}\colon V \to U$ is Lipschitz.*

5.3. **Lemma.** *Let V be an open set in $\mathbf{C}$ with A-uniformly bounded intrinsic distances. Let $g\colon V \to C$ be a function whose distributional first derivatives are bounded measurable functions. Then g is Lipschitz.*

Proof. We start by constructing the standard smoothing of g by convolution. Let ρ be a C^∞ non-negative real valued function on $\mathbf{C}$ with its support contained in the unit disk and with $\int_{\mathbf{C}} \rho(s)\, dV_2(s) = 1$. Set $\rho_\varepsilon(s) := \varepsilon^{-2}\rho(s/\varepsilon)$. Then $\int_{\mathbf{C}} \rho_\varepsilon(s)\, dV_2(s) = 1$ and ρ_ε has its support in the disk of radius ε about the origin. Let $g_\varepsilon(w) = \int_{\mathbf{C}} g(w-s)\rho_\varepsilon(s)\, dV_2(s)$ be the convolution of g and ρ_ε. Letting V_ε be the set of points in V that are a distance of at least ε from the boundary ∂V, g_ε is C^∞ in V_ε and $g_\varepsilon \to g$ uniformly on compact subsets of V as $\varepsilon \to 0$. Convolution commutes with taking distributional partial derivatives, [11, Thm 1.6.1 p. 14], and therefore

$$(g_\varepsilon)_x(w) = \int_{\mathbf{C}} g_x(w-s)\rho_\varepsilon(s)\, dV_2(s), \quad (g_\varepsilon)_y(w) = \int_{\mathbf{C}} g_y(w-s)\rho_\varepsilon(s)\, dV_2(s).$$

By assumption there is a constant C_2 such that $|g_x|, |g_y| \leq C_2$ on V. The formulas for $(g_\varepsilon)_x$ and $(g_\varepsilon)_y$ then show that $|(g_\varepsilon)_x|, |(g_\varepsilon)_y| \leq C_2$ on V_ε. This implies the operator norm of $(g_\varepsilon)'$ satisfies $\|(g_\varepsilon)'(w)\| \leq 2C_2$ on V_ε. Let w_0, w_1 be in V. Then there is a smooth curve $c\colon [0,1] \to V$ with $c(0) = w_0$ and $c(1) = w_1$ and with $\mathrm{Length}(c) \leq A|w_1 - w_2|$. For any ε less than the distance of c from the boundary ∂V we have

$$\begin{aligned} |g_\varepsilon(w_1) - g_\varepsilon(w_0)| &= \left| \int_0^1 \frac{d}{dt} g_\varepsilon(c(t))\, dt \right| \leq \int_0^1 \|(g_\varepsilon)'(c(t)\|\, |c'(t)|\, dt \\ &\leq 2C_2 \,\mathrm{Length}(c) \leq 2C_2 A|w_1 - w_0|. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ gives $|g(w_1) - g(w_0)| \leq 2C_2 A|w_1 - w_0|$ and thus g is Lipschitz as required. □

Proof of Proposition 5.2. Let $z \in U$ and let $r > 0$ be small enough that the disk $B(z,r)$ is contained in U. The restriction of f to $B(z,r)$ will still have bounded distributional first derivatives and $B(z,r)$ is convex, therefore Lemma 5.3 implies that $f\big|_{B(z,r)}$ is Lipschitz. This shows that f is locally Lipschitz on U. Thus by Rademacher's Theorem its derivative $f'(z)$ exists almost everywhere on U. For

a locally Lipschitz function the ordinary first partial derivatives are the some as the distributional first partial derivatives, whence the assumption about f having bounded first distributional derivatives implies there is a constant C_3 such that $\|f'(z)\| \le C_3$ almost everywhere on U. But then $\|f'(z)\|^2 \le (C_3^2/\delta)\delta \le (C_3^2/\delta)J(f)$ almost everywhere. Therefore f is K-quasiconformal with $K = (C_3^2/\delta)$. Calculation shows that the Jacobian is given by $J(f) = |f_z|^2 - |f_{\overline{z}}|^2$ and that $|f_z|, |f_{\overline{z}}| \le \|f'(z)\| \le C_3$. Combining this with Proposition 5.1 yields that the distributional derivatives $(f^{-1})_w$ and $(f^{-1})_{\overline{w}}$ are functions with

$$|(f^{-1})_w| \le \frac{|\overline{f_z}|}{\delta} \le \frac{C_3}{\delta}, \qquad |(f^{-1})_{\overline{w}}| \le \frac{|f_{\overline{z}}|}{\delta} \le \frac{C_3}{\delta}.$$

Therefore the distributional first derivatives of f^{-1} are bounded on V and V has A-uniformly bounded intrinsic distances. Whence Lemma 5.3 implies that f^{-1} is Lipschitz. □

Some basic facts about solutions to the Beltrami equation will also be needed. Let U be a open subset of $\mathbf{C}$ and $\mu\colon U \to \mathbf{C}$ a measurable function with $\|\mu\|_{L^\infty} < 1$. Then the *Beltermi equation* determined by μ is

$$f_{\overline{z}} = \mu f_z.$$

When $\mu \equiv 0$ this is just the Cauchy-Riemann equations. The following summarizes the basic facts about existence and uniqueness of solutions to Beltrami equations and is a combination of a special case of a basic existence result of C. B. Morrey [17] and a factorization theorem of Stoilow. A good source for these results is the book [15] where [15, Thm. 2, p. 8] and [15, Thm. 3, pp. 8–9] can be combined to give:

5.4. **Theorem.** *Let U be a bounded simply connected open subset of $\mathbf{C}$ and $\mu\colon U \to \mathbf{C}$ a measurable function with $\|\mu\|_{L^\infty} < 1$. Let $z_0 \in U$. Then there is a quasiconformal map $q\colon U \to \Delta_r$ that satisfies $q_{\overline{z}} = \mu q_z$ and $q(z_0) = 0$. Moreover, if $f \in W^{1,2}_{\mathrm{Loc}}(U)$ and satisfies $f_{\overline{z}} = \mu f_z$ in the distributional sense in U, then $f(z) = \Phi(q(z))$ for a unique holomorphic function Φ.* □

5.5. **Proposition.** *Let U be an open disk centered at the origin in $\mathbf{R}^2 = \mathbf{C}$ and $f = u + iv$ a Lipschitz function defined on U with $f(0) = 0$, and such that there is a constant $\delta > 0$ with $J(f) = u_x v_y - u_y v_x \ge \delta > 0$ almost everywhere. Then there is an $r > 0$ and a constant $C_o > 0$ such that for any $w \in \mathbf{C}$ with $|w| \le r$ there is a $z \in U$ with $f(z) = w$ and $C_o^{-1}|z| \le |w| \le C_o|z|$.*

Proof. By assumption the Jacobian satisfies $J(f) = |f_z|^2 - |f_{\overline{z}}|^2 \ge \delta$ almost everywhere. This implies $|f_z|^2 \ge \delta + |f_{\overline{z}}|^2 \ge \delta$ and thus $|f_z| \ge \sqrt{\delta} > 0$ almost everywhere. Whence the complex valued function

$$\mu(z) = \frac{f_{\overline{z}}}{f_z}$$

is defined almost everywhere on U. Also $|f_z|^2 - |f_{\overline{z}}|^2 \ge \delta$ implies

$$|\mu(z)|^2 = \left|\frac{f_{\overline{z}}}{f_z}\right|^2 \le 1 - \frac{\delta}{|f_z|^2}.$$

But f is Lipschitz so there is a constant C_4 with $|f_z|^2 \le C_4$ almost everywhere in U. Using this in the last inequality gives

$$|\mu(z)|^2 \le 1 - \frac{\delta}{C_4} := C_5^2 < 1.$$

Thus f satisfies the Beltrami equation $f_{\overline{z}} = \mu(z) f_z$. where $\|\mu\|_{L^\infty} \le C_5 < 1$.

By Theorem 5.4 there is a homeomorphism $q\colon U \to U$ with $q(0) = 0$ and $q \in W^{1,2}_{\mathrm{Loc}}(U)$ that satisfies $q_{\overline{z}} = \mu(z) q_z$ and a holomorphic function Φ defined on U such that $f(z) = \Phi(q(z))$. As $q(0) = 0$ and $f(0) = 0$ the the holomorphic function Φ will have a zero at 0. Assume this zero is of order $k \ge 1$. Then standard results, [2, p. 133], about holomorphic maps imply there is a holomorphic mapping Ψ with $\Psi(0) = 0$, which is conformal near 0, and such that $\Phi(w) = \Psi(w)^k$. Then in a neighborhood of 0 the map $h := \Psi \circ q$ is a homeomorphism and in this neighborhood $f(z) = \Psi(q(z))^k = h(z)^k$. It follows that there is a small positive real number r such that if $\Delta_r := \{w : |w| < r\}$ is the disk of radius r and $\Delta_r^* := \{w : 0 < |w| < r\}$ the pictured disk, and U_r is the connected component of $f^{-1}[\Delta_r]$ containing 0, and $U_r^* = U_r \setminus \{0\}$, then $f\big|_{U_r^*} \colon U_r^* \to \Delta_r^*$ is exactly k to 1, and is in fact a k-fold covering map. (That is each $w \in \Delta_r^*$ has a neighborhood N that is *evenly covered* in the sense that $f\big|_{U_r^*}^{-1}[N]$ is a disjoint union of sets $M_1, \ldots, M_k$ such that $f\big|_{U_r^*}$ restricted to each M_j is a homeomorphism of M_j with N.) Let $f_0 := f\big|_{U_r^*}$. The fundamental groups of Δ_r^* and U_r^* are both isomorphic to the additive group of integers $\mathbf{Z}$ and the image $f_{0*}\left[\pi_1(U_r^*)\right]$ in $\pi_1(\Delta_r^*)$ is $k\mathbf{Z}$, the unique subgroup of index k in $\pi_1(\Delta_r^*)$. Define a map $\varpi\colon \Delta_r^* \to \Delta_r^*$ by

$$\varpi(\rho e^{i\theta}) = \rho e^{ik\theta}.$$

This is also a k-fold covering map and thus $\varpi_*\left[\pi_1(\Delta_r^*)\right]$ also has index k in $\pi_1(\Delta_r^*)$. Whence $\varpi_*\left[\pi_1(\Delta_r^*)\right] = f_{0*}\left[\pi_1(U_r^*)\right]$. Therefore, [16, Thm 5.1, p. 156] or [21, Thm 5, p. 76], there is a continuous lifting $\hat{f}_0\colon U_r^* \to \Delta_r^*$ such that

$$\begin{tikzcd}
 & \Delta_r^* \arrow[d, "\varpi"] \\
U_r^* \arrow[ur, "\hat{f}_0"] \arrow[r, "f_0"] & \Delta_r^*
\end{tikzcd}$$

commutes. Then $f_0 = \varpi \circ \hat{f}_0$ and, [16, Lem. 6.7, p. 160] or [21, Lem. 1, p. 79], $\hat{f}_0$ is also a covering map (which can also easily be checked from the definitions). As $f_0 = \varpi \circ \hat{f}_0$ and both the maps f_0 and ϖ are k to 1 this forces $\hat{f}_0$ to be 1 to 1. But a 1 to 1 covering map is a homeomorphism and thus $\hat{f}_0\colon U_r^* \to \Delta_r^*$ is a homeomorphism.

In polar coordinates (ρ, θ) on Δ_r^* the standard flat Riemannian metric is given by $g_0 := d\rho^2 + \rho^2\, d\theta^2$. The pull back of this metric by ϖ is $\varpi^* g_0 = d\rho^2 + k^2 \rho^2\, d\theta^2$. Therefore $g_0 \le \varpi^* g_0 \le k^2 g_0$. This shows for any vector X and any point $z \in \Delta_r^*$ that $|X| \le |\varpi'(z) X| \le k|X|$. Thus the operator norms of the linear maps $\varpi'(z)$ and $\varpi'(z)^{-1}$ satisfy

$$\|\varpi'(z)\| \le k, \qquad \|\varpi'(z)^{-1}\| \le 1. \tag{5.1}$$

The map $\varpi\colon \Delta_r^* \to \Delta_r^*$ is C^∞ and (5.1), together with the inverse function theorem, shows that each point $w \in \Delta_r^*$ has a neighborhood N such that $\varpi\big|_N$ is injective, $\varpi[N]$ is a open subset of Δ_r^*, and $\varpi\big|_N$ is a diffeomorphism of N with $\varpi[N]$. Let $z_0 \in U_r^*$ and let N be such a neighborhood of $w_0 = \hat{f}_0(z_0)$. The point z_0 will have a neighborhood V such that $f_0(z) \in \varpi[N]$ for all $z \in V$. Thus $f_0 = \varpi \circ \hat{f}_0$ implies $\hat{f}_0\big|_V = \varpi\big|_N^{-1} \circ f_0\big|_V$. The function $\varpi\big|_N^{-1}$ is C^∞ and f_0 is Lipschitz, thus $\hat{f}_0$ is Lipschitz near z_0. Therefore $\hat{f}_0'$ exists almost everywhere in V and for $z \in V$

where $\hat{f}_0'(z)$ exists use (5.1) to get

$$\|\hat{f}_0'(z)\| \leq \left\| \left(\varpi^{-1}\big|_N^{-1}\right)' (f_0(z)) \right\| \|f_0'(z)\| \leq \|f_0'(z)\|.$$

But this holds in a neighborhood of an arbitrary point z_0 of U_r^* and whence $\|\hat{f}_0'(z)\| \leq \|f_0'(z)\|$ almost everywhere on U_r^*. As f_0 is Lipschitz there is a constant C_6 such that $\|\hat{f}_0'(z)\| \leq \|f_0'(z)\| \leq C_6$ almost everywhere on U_r^*. This shows that the distributional first derivatives of $\hat{f}_0$ are bounded measurable functions.

It is easy to compute that $J(\varpi) = k$. By assumption, $J(f) \geq \delta$ and f_0 is a restriction of f whence

$$\delta \leq J(f_0) = J(\varpi \circ \hat{f}_0) = J(\varpi)J(\hat{f}_0) = kJ(\hat{f}_0).$$

Therefore $J(\hat{f}_0) \geq \delta/k$. The set Δ_r^* has A-uniformly bounded intrinsic distances for all $A > 1$. Thus $\hat{f}_0 \colon U_r^* \to \Delta_r^*$ satisfies the conditions of Proposition 5.2. Whence $\hat{f}_0^{-1} \colon \Delta_r^* \to U_r^*$ is Lipschitz.

As $\hat{f}_0^{-1}$ is Lipschitz there is a constant C_7 such that for all $w, w_0 \in \Delta_r^*$ the inequality $|\hat{f}_0^{-1}(w) - \hat{f}_0^{-1}(w_0)| \leq C_7|w - w_0|$ holds. Therefore for any $z, z_0 \in U_r^*$

$$|z - z_0| = |\hat{f}_0^{-1}(\hat{f}_0(z)) - \hat{f}_0^{-1}(\hat{f}_0(z_0))| \leq C_7|\hat{f}_0(z) - \hat{f}_0(z_0)|. \tag{5.2}$$

From the definition of ϖ it clear that $|\varpi(w)| = w$ for all $w \in \Delta_r^*$. Thus $|\hat{f}_0(z_0)| = |\varpi(f_0(z_0))| = |f_0(z_0)| = |f(z_0)|$. But $f(0) = 0$ and f is continuous and whence $\lim_{z_0 \to 0} f(z_0) = 0$. Therefore $\lim_{z_0 \to 0} \hat{f}(z_0) = 0$ and thus taking the limit as $z_0 \to 0$ in (5.2) yields

$$|\hat{f}_0(z)| \geq \frac{1}{C_7}|z|$$

for all $z \in U_r^*$.

We now complete the proof of Proposition 5.5. Let $w \in \Delta_r^*$. Then there is a $z \in U_r^*$ with $f(z) = w$. By the definition of f_0 as the restriction of f we have $w = f_0(z) = \varpi(\hat{f}_0(z))$. Again using that $|\varpi(\xi)| = |\xi|$ we have

$$|w| = |\varpi(\hat{f}_0(z))| = |\hat{f}_0(z)| \geq \frac{1}{C_7}|z|.$$

Also, as f is Lipschitz and $f(0) = 0$, there is a constant C_8 with $|w| = |f(z)| \leq C_8|z|$. Letting $C_o = \max\{C_7, C_8\}$ completes the proof. □

5.2. **Proof of the Main Lemma.** We use the notation of the Section 4. In particular $\varphi(u) = p(u)u + \nabla p(u)$, ψ is the projection of φ onto the first two coordinates and $u = u(x, y)$ is given by (4.7).

5.6**. Lemma.** *There is an open disk U centered at the origin so that for some constant $\delta > 0$ the Jacobian of ψ satisfies $J(\psi) := \det(\psi') \leq -\delta$ almost everywhere in U.*

Proof. For (x, y) in the unit disk the tangent plane to $\mathbb{S}^2$ at $u(x, y)$ is $u(x, y)^\perp$ and the orientation of this tangent plane is so that the projection onto the (x, y) plane is orientation preserving. (This is because $u(x, y)$ is in the upper hemisphere of $\mathbb{S}^2$.) By Proposition 3.2 $\varphi'(z) = p(z)I + \nabla^2 p(z)$ almost everywhere and by Lemma 4.3

$$J(\varphi) = \det(p(z)I + \nabla^2 p(z)) \leq -\delta_0$$

for almost all z in the unit disk and for some $\delta_0 > 0$. The projection π of the tangent plane $T(\mathbb{S}^2)_u = u^\perp$ onto $\mathbf{R}^2$ has Jacobian $J(\pi) = \langle u, e_3\rangle$. As $\psi = \pi \circ \varphi$

$$J(\psi) = J(\pi)J(\varphi) = \langle u, e_3\rangle J(\varphi) \leq -\langle u, e_3\rangle \delta.$$

But $\langle u(x,y), e_3\rangle = \sqrt{1-x^2-y^2}$ so if $U = \{(x,y) : x^2+y^2 < \sqrt{3}/2\}$, then $J(\pi) > 1/2$. Thus on U $J(\psi) < -\delta$ where $\delta = \frac{1}{2}\delta_0$. □

Returning to the proof of the Main Lemma, let U be as in the last lemma and let $f\colon U \to \mathbf{C}$ be given by

$$f(z) = \overline{\psi(z)} - \overline{\psi(0)}. \tag{5.3}$$

Complex conjugation is an orientation reversing isometry and ψ is Lipschitz, thus f is also Lipschitz. The Jacobian of f is $J(f) = -J(\psi) \geq \delta$. And clearly $f(0) = 0$. Note that as f and ψ are related by (5.3), then $\psi(z) = w$ if and only if $f(z) = \overline{w} - \overline{\psi(0)}$. Therefore the Main Lemma 4.5 follows from Proposition 5.5. This completes the proof of Theorem 4.1.

Acknowledgments

I am indebted to Daniel Hug for supplying the statement and a proof of Proposition 2.3 which greatly simplified my initial proof of the $C^{1,1}$ regularity of the support function of a set of constant width. A remark of Marek Kossowski lead me to realize the covering space argument in the proof of the Proposition 5.5 was required. I also had several useful conversations with Mohammad Ghomi on topics related to this paper.

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